Last Revised: 08/23/2023

2024 Level 1 - Quantitative Methods

Interest Rates, Present Value, and Future Value

a. interpret interest rates as required rates of return, discount rates, or opportunity costs

b. explain an interest rate as the sum of a real risk-free rate and premiums that compensate investors for bearing distinct types of risk

c. calculate and interpret the future value (FV) and present value (PV) of a single sum of money, an ordinary annuity, an annuity due, a perpetuity (PV only), and a series of unequal cash flows

d. demonstrate the use of a time line in modeling and solving time value of money problems

e. calculate the solution for time value of money problems with different frequencies of compounding

f. calculate and interpret the effective annual rate, given the stated annual interest rate and the frequency of compounding

Time Value of Money

Page 1
LOS a
- interpret

- 3 rules of money: 1. money sooner is worth more than money later
  2. larger cash flows are worth more than smaller cash flows
  3. less risky cash flows are worth more than risky cash flows

- Interest rates (r) - can be thought of in 3 ways:

1/ required rate of return → the rate of return required by an investor or lender → moneytoday (1 + r) = moneytomorrow

2/ discount rate → the rate at which some future value is discounted to arrive at a value today → $\frac{\mathbf{moneytomorrow}}{(\mathbf{1+r})} = \mathbf{moneytoday}$

3/ opportunity cost → the value an investor or lender forgoes by choosing a particular action
i.e. r is the opportunity cost of current consumption

- typically: req. rate of return = discount rate = opportunity cost

Page 2
LOS b
- explain

→ suppose I lend you $1000 for one year. I will want:

r_f → real risk-free rate: single period rate

\+ inflation premium → compensates for expected inflation (π^e)

$r_f + \pi^e$ = nominal risk-free rate
$[(\mathbf{1}+\mathbf{r_f})(\mathbf{1}+\boldsymbol{\pi}^{\mathbf{e}})]-\mathbf{1}$

\+ Default risk premium - compensates for credit risk

\+ Liquidity premium → risk of loss vs. fair value if an investment needs to be converted to cash quickly

\+ Maturity premium → greater interest rate risk (i.e. price risk) with longer maturities
→ will also have a premium for inflation uncertainty → the longer the time period, the more uncertain we are about the level of expected inflation

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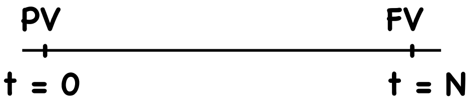
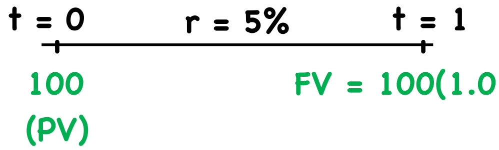
LOS c
- calculate
- interpret

→ Future Value of a Single Cash Flow/

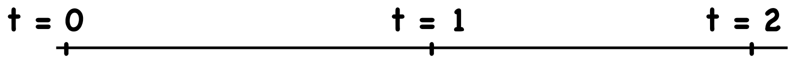


PV = present value
FV = future value
r = interest rate
N = # of periods
} r must be in the same periodicity as N

e.g./ \$100, 5yrs., 6% → $100(1.06)^5$
semi-annual → $100(1.03)^{10}$
quarterly → $(1.015)^{20}$



FV = 100(1.05) = 105



100 100(1.05) $\underbrace{100(1.05)(1.05) = 100(1.05)^2}_{FV = PV\ (1 + r)^N} = 110.25$

- the power of compounding/

S1

at 5%, N = 40: FV = $1000(1.05)^{40}$ = 7,039.99. (2000)

at 7%, N = 40: FV = $1000(1.07)^{40}$ = 14,974.46 (2800)

at 9%, N = 40: FV = $1000(1.09)^{40}$ = 31,409.42. (3600)

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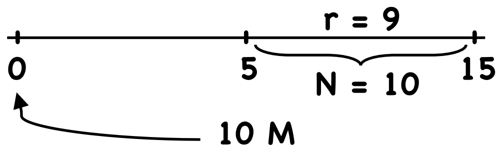
LOS c
- calculate
- interpret

e.g./ \$5M at t = 0, r = 7% compounded annually, N = 5 years

Find FV: method 1: $5M(1.07)^5$ = 7,012,758.65
method 2: N = 5, I/Y = 7, PV = -5,000,000, PMT = 0
CPT FV = 7,012,758.65

e.g. 2/ Invest ¥2.5M, r = 8% compounded annually, N = 6 years
- Find FV: method 1: $2.5M(108)^6$ = ¥3,967,186
method 2: N = 6, I/Y = 8, PMT = 0, PV = -2,500,000
CPT FV = 3,967,186

e.g. 3/ \$10M at t = 5, E(r) = 9%, FV at t = 15?



method 1: FV = $10M(1.09)^{10}$ = 23,673,636.75
method 2: N = 10, I/Y = 9, PMT = 0
PV = -10,000,000
CPT FV = 23,673,636.75

$$PV_0 = \frac{10,000,000}{(1.09)^5} = 6,499,313.86$$

Page 5
LOS e
- solve

→ Frequency of Compounding/

- all rates are quoted annually → r_s (stated interest rate)

e.g./ PV = 10,000, N = 2, r_s = 8% compounded quarterly, FV = ?

$$\mathrm{FV} = \mathrm{PV}\left(1 + {r_s}/_{m}\right)^{mN} = 10,000\left(1 + {.08}/_{4}\right)^{4\times 2} = 10,000(1.02)^8 = 11,716.59$$

or/

N = 2 × 4 = 8, I/Y = $^8/_4$ = 2, PMT = 0, PV = -10,000, CPT FV = 11,716.59

e.g. 2/ PV = $1M, N = 1, r = 6% compounded monthly, FV = ?

$$\mathrm{FV} = 1M\left(1 + {.06}/_{12}\right)^{12\times 1} = 1M(1.005)^{12} = 1,061,678.81$$

or/ N = 1 × 12 = 12, I/6 = $^6/_{12}$ = .5, PMT = 0, PV = -1,000,000

CPT FV = 1,061,678.81

→ Continuous Compounding/ $\mathrm{FV} = \mathrm{PV}_{e}^{r_s \times N}$

e.g./ PV = 10,000, N = 2, r = 8% compounded continuously, FV = ?

$$\mathrm{FV} = 10,000e^{.08 \times 2} = 11,735.11$$

or/ .08 × 2 = 2nd LN × 10 000 = 11,735.11

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LOS f
- calculate
- interpret

→ EAR: effective annual rate

			EAR
$100 at 8%	annual	100(1.08) = 108	8%
	semi-annual	$100(1.04)^2$ = 108.16	8.16%
	quarterly	$100(1.02)^4$ = 108.2432	8.2432%
	monthly	$100(1.00\dot{6})^{12}$ = 108.30	8.3%
	daily	$100(1.000219)^{365}$ = 108.3278	8.3278%
	continuous	$100e^{.08}$ = 108.3287	8.3287%

$$\underbrace{\qquad\qquad}_{\left[\left(1 + {r}/_{m}\right)^{mN} - 1\right]}$$

or $(e^{rN} - 1)$

- if we know EAR, we can solve for r_s

e.g./ EAR of 8.3%, compounded monthly

$$.083 = \left(1 + {r_s}/_{12}\right)^{12} - 1$$

$$1.083 = \left(1 + {r_s}/_{12}\right)^{12}$$

$$(1.083)^{1/12} = 1 + {r_s}/_{12}$$

$$(1.083)^{1/12} - 1 = {r_s}/_{12}$$

$$12\left[(1.083)^{1/12} - 1\right] = r_s$$ → 1.083 y^x (1 ÷ 12) - 1 × 12 =

$$r_s = 8\%$$

e.g. 2/ .083287 = $e^{r_s} - 1$

1.083287 = e^{r_s}

LN (1.083287) = r_s → r_s = 8%

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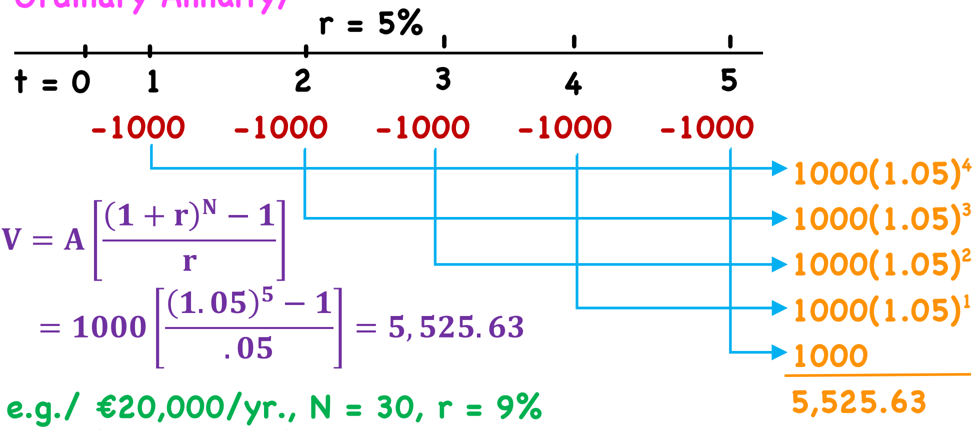
LOS c
- calculate
- interpret

→ Future value of a series of cash flows:

annuity - a finite set of level sequential cash flows
↳ ordinary → first cash flow at t = 1
↳ due → first cash flow at t = 0

→ Ordinary Annuity/



$$FV = A\left[\frac{(1+r)^N - 1}{r}\right]$$

$$= 1000\left[\frac{(1.05)^5 - 1}{.05}\right] = 5,525.63$$

N = 5
PMT = -1000
I/Y = 5
PV = 0
CPT FV

e.g./ $\underbrace{\text{€20,000/yr.,}}_{\text{end of yr. CF}}$ N = 30, r = 9%

- ordinary annuity/

$$= 20,000\left[\frac{(1.09)^{30} - 1}{.09}\right] = 2,726,150.77$$

or/ N = 30, PMT = -20,000, PV = 0, I/Y = 9
CPT FV

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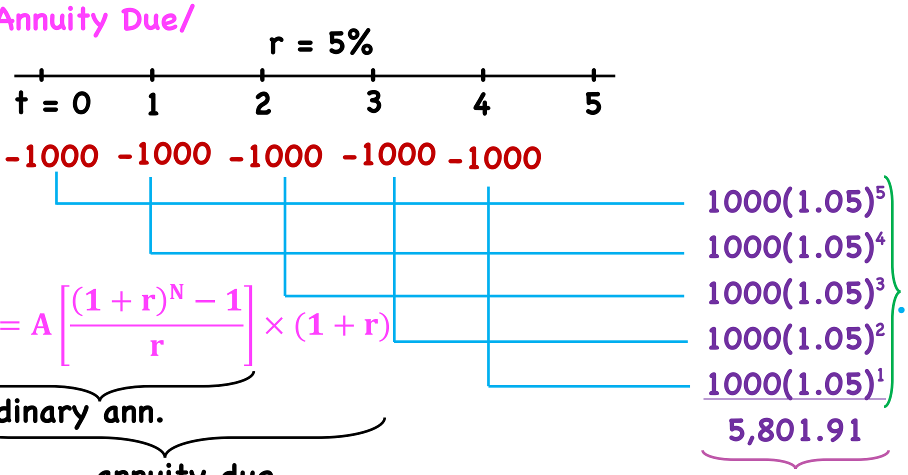
LOS c
- calculate
- interpret

→ Future value of a series of cash flows:

• Annuity Due/



= ordinary annuity × (1.05)

• calculator in END mode
N = 5
PMT = -1000
PV = 0
I/Y = 5

$$\underbrace{\underbrace{FV = A\left[\frac{(1+r)^N - 1}{r}\right]}_{\text{ordinary ann.}} \times (1+r)}_{\text{annuity due}}$$

calculator in BGN mode
2nd PMT 2nd ENTER CE/C

Display END BGN.
N = 5, PMT = -1000, PV = 0, I/Y = 5
CPT FV

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LOS c
- calculate
- interpret

→ Future value of a series of cash flows:

- Unequal cash flows

r = 5%

t = 0 | 1 | 2 | 3 | 4 | 5

-1000 × $(1.05)^4$ -2000 × $(1.05)^3$ -4000 × $(1.05)^2$ -5000 × $(1.05)^1$ -6000 × $(1.05)^0$ = 19,190.76

→ Present value of a single cash flow:

r = 8%

t = 0   t = 6

PV = ? ← 100,000

$$PV = {}^{100,000}\!/_{(1.08)^6} = 63,019.96$$

(N = 6, PMT = 0, I/Y = 8, FV = 100,000)
CPT PV

$FV = PV\,(1 + r)^N$

$$PV = \frac{FV}{(1+r)^N}$$

$= FV\,(1 + r)^{-N}$

e.g./

r = 8%

t = 0 | 4 | 8

10   100,000

$$PV_4 = {}^{100,000}\!/_{(1.08)^6}$$

$$= 63,016.96$$

$$PV_0 = {}^{PV_4}\!/_{(1.08)^4}$$

$$= 46,319.35$$

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LOS c
- calculate
- interpret

→ Present value of a series of cash flows:

- Ordinary annuity/

1000   1000   1000   1000   1000

t = 0 | 1 | 2 | 3 | 4 | 5

r = 12%

$$PV = A\left[\frac{1 - \frac{1}{(1+r)^N}}{r}\right] \rightarrow 1000\left[\frac{1 - \frac{1}{(1.12)^5}}{.12}\right] = 3,604.78$$

N = 5
or/ FV = 0
I/Y = 12
PMT = 1000   CPT PV

- Annuity Due

1000   1000   1000   1000   1000

t = 0 | 1 | 2 | 3 | 4 | 5

$$1000 + 1000\left[\frac{1 - \frac{1}{(1.12)^4}}{.12}\right] = 4037.35$$

or/ N = 4
PMT = 1000
I/Y = 12
FV = 0
CPT PV
+1000

e.g./

200k   200k   200k   200k

t = 0 | 1 | 2 | 19 | 20

R = 7%

$2M today or 20 PMTs?

N = 19, PMT = 200,000, I/Y = 7, FV = 0
CPT PV (+ 20,000) = 2,267,119.05

or/

$$200k + 200k\left[\frac{1 - \frac{1}{(1.07)^{19}}}{.07}\right] = 2,267,119.05$$

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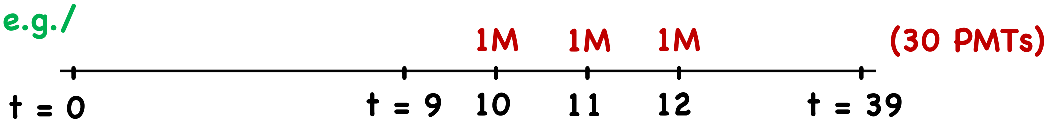
LOS c
- calculate
- interpret

→ Present value of a series of cash flows:

e.g./



PV₀ = ?

r = 5%

PV_{10} → annuity due

PV_9 → ordinary annuity

$$PV_{10} = 1M\left[\frac{1-\frac{1}{(1.05)^{29}}}{.05}\right] + 1M$$

$$= 16,141,073.58$$

$$PV_0 = {}^{PV_{10}}/_{(1.05)^{10}} = 9,909,219$$

$$PV_9 = 1M\left[\frac{1-\frac{1}{(1.05)^{30}}}{.05}\right] = 15,372,451.03$$

$$PV_0 = {}^{PV_9}/_{(1.05)^9} = 9,909,219$$

N = 30, PMT = 1,000,000, I/Y = 5, FV = 0
CPT PV_9
↓
$PV_0 = {}^{PV_9}/_{(1.05)^9}$

BGN mode
N = 30, PMT = 1,000,000, I/Y = 5, FV = 0
CPT PV_{10}
↓
$PV_0 = {}^{PV_{10}}/_{(1.05)^{10}}$

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LOS c
- calculate
- interpret

→ Present value of a $\underbrace{\text{perpetuity}}$: $PV = {}^{A}/_{r}$

- level CFs
- sequential
- infinite

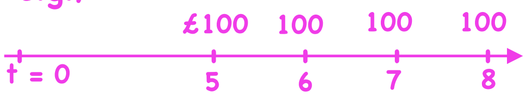
e.g./ $10 at t = 1 forever
r = 20%

$$PV = {}^{10}/_{.20} = \$50$$

e.g./ £ 100/yr. - perpetual, r = 5%

$$PV = {}^{100}/_{.05} = £2,000$$

e.g./



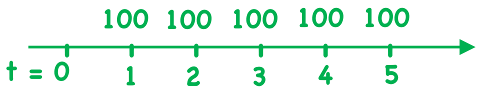
(r = 5%)

$$PV_5 = {}^{100}/_{.05} + 100 = 2100 \qquad PV_0 = \frac{2100}{(1.05)^5} = 1645.40$$

$$PV_4 = {}^{100}/_{.05} = 2000 \qquad PV_0 = \frac{2000}{(1.05)^4} = 1645.40$$

e.g./

long



perpetuity at t = 0 $\quad PV_0 = {}^{100}/_{.05} = 2,000$

short

perpetuity at t = 4 $\quad P_0 = {}^{PV_4}/_{(1.05)^4} = \frac{1645.40}{354.60}$

= 4 yr. ord. annuity

$$100\left[\frac{1-\frac{1}{(1.05)^4}}{.05}\right] = 354.60$$

N = 4
PMT = 100
I/Y = 5 CPT PV
FV =0

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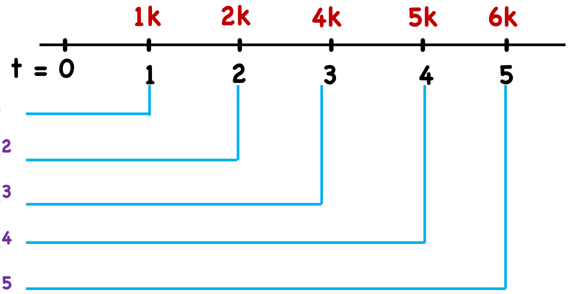
LOS c
- calculate
- interpret

→ Present value of a series of unequal CFs:



$1000/(1.05)$
$2000/(1.05)^2$
$4000/(1.05)^3$
$5000/(1.05)^4$
$6000/(1.05)^5$
$= 15{,}036.46$

Calculator:

2nd	CF	2nd	CE/C
CF_0		↓	
CO1	1000	ENTER	↓ ↓
CO2	2000	ENTER	↓ ↓
CO3	4000	ENTER	↓ ↓
CO4	5000	ENTER	↓ ↓
CO5	6000	ENTER	↓ NPV
I	5	ENTER	
NPV		CPT	

15,036.46

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LOS c
- calculate
- interpret

→ Solve for r, N or PMT/

$FV = PV(1 + r)^N$ → solve for r

$$FV/PV = (1+r)^N$$
$$(FV/PV)^{1/N} = 1 + r$$
$$r = (FV/PV)^{1/N} - 1$$

e.g./

Year	Sales	Profit
2008	10,503	822.5
2012	14,146.4	796.4

- growth in Sales ⇒ $14{,}146.40 = 10{,}503\ (1 + g)^4$

$$g = \left(14146.40/10503\right)^{1/4} - 1 = .07729 \quad (7.73\%)$$

- growth in Profit ⇒ $796.4 = 822.5\ (1 + g)^4$

$$g = \left(796.4/822.5\right)^{1/4} - 1 = -.00803 \quad (-.803\%)$$

e.g./ 2012 - 7.35M units sold
2007 - 8.52M units sold
find g

$$g = (FV/PV)^{1/N} - 1$$
$$= \left(7.35/8.52\right)^{1/5} - 1 = -.02911 \quad (-2.911\%)$$

Note: g is called the compound annual growth rate

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LOS c
- calculate
- interpret

→ Solve for r, N or PMT/

e.g./ How long will it take to double €10M at 7% compounded annually?

→ Solve for N: $\mathbf{FV = PV(1+r)^N}$

$$(1+r)^N = {}^{FV}/_{PV}$$

$$N\,\text{In}(1+r) = \text{In}\left({}^{FV}/_{PV}\right)$$

$$N = \frac{\text{In}\left({}^{FV}/_{PV}\right)}{\text{In}(1+r)}$$

$$20M = 10M\,(1.07)^N$$

$$N = \frac{\text{In}\left({}^{20}/_{10}\right)}{\text{In}(1.07)} = \frac{.69314}{.06765} = 10.244$$

calculator: 20 ÷ 10 = LN ÷ (1.07 LN) =

→ solve for PMT: $100,000 mortgage, 30 yrs., 8% compounded monthly

$$PV = A\left[\frac{1-\frac{1}{(1+r)^N}}{r}\right]$$

$$A = \frac{PV}{\left[\frac{1-\frac{1}{(1+r)^N}}{r}\right]} = \frac{100,000}{\left[\frac{1-\frac{1}{\left(1+{}^{.08}/_{12}\right)^{12\times30}}}{{}^{.08}/_{12}}\right]} = \frac{100,000}{136.2834} = 733.76$$

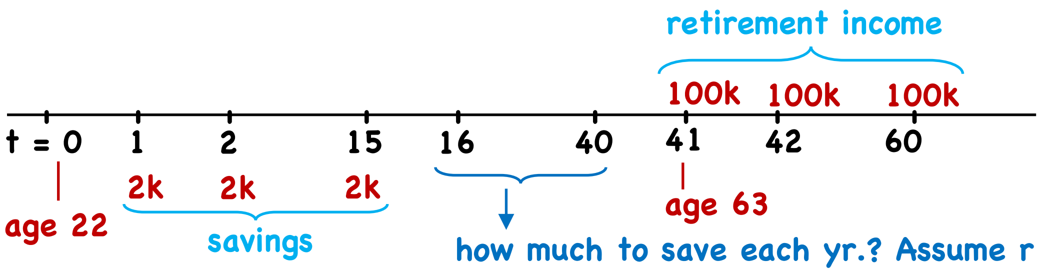
↓

666.66 interest

N = 360
PV = -100,000
FV = 0
I/Y = 8/12 = .66$\dot{6}$

CPT PV

Page 16
LOS d
- demonstrate



how much to save each yr.? Assume r = 8%

(PMT = ?)

1. FV_{15} → N = 15, PMT = 2000, I/Y = 8, PV = 0 CPT FV = 54,304.23
2. PV_{40} → N = 20, PMT = 100,000, I/Y = 8, FV = 0 CPT PV = 981,814.74
3. PV_{15} → 143,362.53 - 54,304.23 = 89,058.30

N = 25, FV = 0, I/Y = 8, PV = -89,058.30
CPT PMT = 8,342.87

$$PV_{15} = {}^{PV_{40}}/_{(1.08)^{25}} = 143,362.53$$

or/

$PV_{40} = PV_{15}(1.08)^{25} = 371,901.17$
$FV_{40} = 981,814.74 - 371,901.17 = 609,913.56$

N = 25, FV = 609,913.56, I/Y = 8
PV = 0 CPT PMT = 8342.87