

LM04 Probability Trees and Conditional Expectations

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1. Introduction

Since many investment decisions are made in an environment of uncertainty, it is essential for portfolio managers and investment managers to have a fundamental grasp of probability concepts. In this learning module, we will focus on:

- Calculation of the expected value, variance, and standard deviation for a random variable.
- Using probability trees to visualize conditional expectations and the total probabilities for expected value.
- Using Bayes' formula to adjust probabilities with the arrival of new information.

Instructor's Note: Before we get into the actual concepts covered in the curriculum, some prerequisite fundamental concepts are presented below.

Fundamental Concepts

A *random variable* is an uncertain quantity/number. For example, when you roll a die, the result is a random variable.

An *outcome* is the observed value of a random variable. For example, if you roll a 2, it is an outcome.

An *event* can be a single outcome or a set of outcomes. For example, you can define an event as rolling a 2 or rolling an even number.

Mutually exclusive events are events that cannot happen at the same time. For example, rolling a 2 and rolling a 3 are examples of mutually exclusive events. They cannot happen at the same time.

Exhaustive events are those that cover all possible outcomes. For example, 'rolling an even number' or 'rolling an odd number' are exhaustive events. They cover all possible outcomes.

The *two defining properties of probability* are:

- The probability of any event has to be between 0 and 1.
- The sum of the probabilities of mutually exclusive and exhaustive events is equal to 1.

Conditional v/s Unconditional probabilities

Unconditional probability is the probability of an event occurring irrespective of the occurrence of other events. It is denoted as $P(A)$. Unconditional probability is also called 'marginal' probability.

Conditional probability is the probability of an event occurring given that another event has occurred. It is denoted as $P(A|B)$, which is the probability of event A given that event B has occurred.

Joint Probability and Multiplication Rule

Multiplication rule is used to determine the joint probability of two events. It is expressed as:

$$P(AB) = P(A|B) P(B)$$

Rearranging the equation, we get the formula for computing conditional probabilities:

$$P(A|B) = P(AB) / P(B)$$

Example

$P(\text{interest rates will decrease}) = P(D) = 40\%$

$P(\text{stock price increases}) = P(S)$

$P(\text{stock price will increase given interest rates decrease}) = P(S|D) = 70\%$

Compute probability of a stock price increase **and** an interest rate decrease.

Solution:

$$P(SD) = P(S|D) \times P(D) = 0.7 \times 0.4 = 0.28 = 28\%$$

Addition Rule for Probabilities

Addition rule is used to determine the probability that at least one of the events will occur. It is expressed as:

$$P(A \text{ or } B) = P(A) + P(B) - P(AB)$$

$P(AB)$ represents the joint probability that both A and B will occur. It is subtracted from the sum of the unconditional probabilities: $P(A) + P(B)$, to avoid double counting.

If the two events are mutually exclusive, the joint probability: $P(AB)$ is zero and the probability that either A or B will occur is simply the sum of the unconditional probabilities for each event:

$$P(A \text{ or } B) = P(A) + P(B)$$

Example

$P(\text{price of A increases}) = P(A) = 0.5$

$P(\text{price of B increases}) = P(B) = 0.7$

$P(\text{price of A and B increases}) = P(AB) = 0.3$

Compute the probability that the price of stock A **or** the price of stock B increases.

Solution

$$P(A \text{ or } B) = 0.5 + 0.7 - 0.3 = 0.9$$

Independent and Dependent Events

If the occurrence of one event does not influence the occurrence of the other event, then the two events are called *independent events*.

i.e. $P(A|B) = P(A)$ or $P(B|A) = P(B)$

Multiplication rule for independent events: $P(AB) = P(A) P(B)$

Addition rule for independent events: $P(A \text{ or } B) = P(A) + P(B) - P(AB)$. (The addition rule does not change.)

If the probability of an event is affected by the occurrence of another event, then it is called a *dependent event*.

Total Probability Rule

The total probability rule is used to calculate the unconditional probability of an event, given conditional probabilities.

In investment analysis, we often formulate a set of mutually exclusive and exhaustive scenarios and then estimate the probability of a particular event. For example, let's say that we have two scenarios S and non-S that are mutually exclusive and exhaustive.

According to the total probability rule, the probability of any event P(A) can be expressed as:

$$P(A) = P(AS) + P(AS^c)$$

Using the multiplication rule we get,

$$P(A) = P(A|S) P(S) + P(A|S^c) P(S^c)$$

If we have more than two scenarios, we can generalize this equation to:

$$P(A) = P(AS_1) + P(AS_2) + \dots + P(AS_n) = P(A|S_1) P(S_1) + P(A|S_2) P(S_2) + \dots + P(A|S_n) P(S_n)$$

2. Expected Value and Variance

Expected Value of a Random Variable

The expected value of a random variable can be defined as the probability-weighted average of the possible outcomes of the random variable. For a random variable X, the expected value of X is denoted as E(X) and is calculated as:

$$E(X) = \sum_{i=1}^n P(X_i) X_i$$

where:

X_i = One of n possible outcomes of the random variable X

$P(X_i)$ = Probability of X_i

Variance of a Random Variable

The expected value is our forecast, but we cannot count on the individual forecast being realized. This is why we need to measure the risk we face. Variance and standard deviation are examples of how we can measure this risk.

The variance of a random variable is the expected value (the probability-weighted average) of squared deviations from the random variable's expected value:

$$\sigma^2(X) = E [X - E(X)]^2.$$

It can also be expressed as:

$$\sigma^2(X) = \sum_{i=1}^n P(X) [X - E(X)]^2$$

Variance is a number greater than or equal to 0 because it is the sum of squared terms. If variance is 0, there is no dispersion or risk. The outcome is certain and the quantity X is not random at all.

Standard deviation is the positive square root of variance. Like variance, standard deviation also measures dispersion, but it is measured in the same units as the variable.

We can calculate the expected value and variance of a random variable using a financial calculator as shown below:

Example

A project's cash flow for the upcoming year depends on the state of the economy, as shown in the table below. What is the expected value of the cash flow? What is the variance of the cash flow? What is the standard deviation?

State of Economy	Probability	Cash Flow
Good	0.3	50
Average	0.5	40
Weak	0.2	20

Solution:

Using a financial calculator:

Keystrokes	Explanation	Display
[2nd] [DATA]	Enters data entry mode	
[2nd] [CLR WRK]	Clears data register	X01
50 [ENTER]	1 st possible value of random variable	X01 = 50
[↓] 30 [ENTER]	Probability of 30% for X01	Y01 = 30
[↓] 40 [ENTER]	2 nd possible value of random variable	X02 = 40
[↓] 50 [ENTER]	Probability of 50% for X02	Y02 = 50
[↓] 20 [ENTER]	3 rd possible value of random variable	X03 = 20

[↓] 20 [ENTER]	Probability of 20% for X03	Y03 = 20
[2nd] [STAT]	Puts calculator into stats mode	
[2nd] [SET]	Press repeatedly till you see →	1-V
[↓]	Total number of entries	N = 100
[↓]	Expected value of random variable	X = 39
[↓]	Sample standard deviation	Sx = 10.49
[↓]	Population standard deviation	$\sigma_x = 10.44$

We can then square the population standard deviation of 10.44 to get the variance i.e. $10.44^2 = 109.00$

3. Probability Trees and Conditional Expectations

Total Probability Rule for Expected Value

Just like the total probability rule states unconditional probabilities in terms of conditional probabilities, the total probability rule for expected values states unconditional expected values in terms of conditional expected values.

$$E(X|S) = P(X_1|S) X_1 + P(X_2|S) X_2 + \dots + P(X_n|S) X_n$$

Instructor's Note

Notice that this formula is exactly similar to the total probability rule formula.

$$P(A) = P(A|S_1) P(S_1) + P(A|S_2) P(S_2) + \dots + P(A|S_n) P(S_n)$$

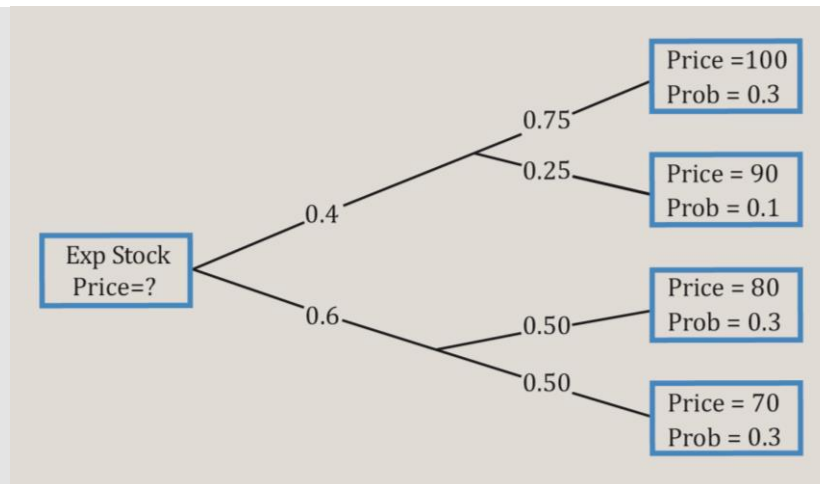
A probability tree is a means of illustrating the results of two or more independent events. The following example illustrates how we can use probability trees and the total probability rule to calculate the expected value of a random variable.

Example

What is the expected price of a stock at the end of the current period given the following information: probability that interest rates will decline = 0.4. If interest rates decline there is a 75% chance that stock price will be \$100 versus a 25% chance that the stock price will be \$90. If interest rates do not decline there is a 50% chance that the stock price will be \$80 versus a 50% chance that stock price will be \$70.

Solution:

We can plot the probabilities using a tree diagram.



Consider the first node (top right). It refers to the probability that the stock price will be \$100 given a decline in interest rates. We can calculate the probability of that happening by multiplying the probability of a decline in interest rates (0.4) by the probability of the stock price being \$100 if that happens (0.75). This gives us a conditional probability of 0.30. In short, it is the joint probability of the stock price being \$100 given a decline in interest rates. Similarly, probabilities are calculated for each of the other three nodes. We can then calculate:

$$E(\text{Price} \mid \text{decline in interest rates}) = 0.75 (\$100) + 0.25 (\$90) = \$97.50$$

$$E(\text{Price} \mid \text{no decline in interest rates}) = 0.50 (\$80) + 0.50 (\$70) = \$75.00$$

Now we use the total probability rule for expected value of stock price at the end of the current period:

$$E(\text{Price}) = E(\text{Price} \mid \text{decline in interest rates}) P(\text{decline in interest rates}) + E(\text{Price} \mid \text{no decline in interest rates}) P(\text{no decline in interest rates})$$

$$E(\text{Price}) = \$97.50 (0.40) + \$75.00 (0.60)$$

$$E(\text{Price}) = \$84.00$$

4. Bayes' Formula and Updating Probability Estimates

Bayes' formula is a rational method for updating or adjusting the probability of an event based on new information. According to Bayes' formula, the updated probability of an event given new information is:

$$P(\text{Event} \mid \text{Information}) = \frac{P(\text{Information} \mid \text{Event})}{P(\text{Information})} \times P(\text{Event})$$

Example

Consider a factory that has three assembly lines. The percentage of output produced at each assembly line is as follows: Line A = 45%, Line B = 35%, Line C = 20%. The output defective from each line is estimated to be 3%, 5%, and 4%, respectively. Given that the product is defective, what is the probability that it came from Line C?

Solution:

When dealing with questions related to Bayes' formula, the first step is to reproduce the information in probability notation:

$$P(\text{Line A}) = 0.45; P(\text{Not Line A}) = 0.55$$

$$P(\text{Line B}) = 0.35; P(\text{Not Line B}) = 0.65$$

$$P(\text{Line C}) = 0.20; P(\text{Not Line C}) = 0.80$$

$$P(\text{Defective} | \text{Line A}) = 0.03, P(\text{Defective} | \text{Line B}) = 0.05, P(\text{Defective} | \text{Line C}) = 0.04$$

$$P(\text{Defective}) = 0.45 \times 0.03 + 0.35 \times 0.05 + 0.20 \times 0.04 = 0.039$$

Next write down the Bayes formula:

$$P(\text{Event} | \text{Information}) = \frac{P(\text{Information} | \text{Event})}{P(\text{Information})} \times P(\text{Event})$$

We then have to distinguish between the event and the information and plug the relevant values into the formula. In this case, the information is that the product is defective. Hence, the formula can be written as:

$$P(\text{Line C} | \text{Defective}) = \frac{[P(\text{Defective} | \text{Line C}) * P(\text{Line C})]}{P(\text{Defective})} = \frac{0.04 * 0.20}{0.039} = 20.51\%$$

Summary

LO: Calculate expected values, variances, and standard deviations and demonstrate their application to investment problems.

The expected value of a random variable can be defined as the probability-weighted average of the possible outcomes of the random variable.

$$E(X) = \sum_{i=1}^n P(X_i) X_i$$

The variance of a random variable is the expected value (the probability-weighted average) of squared deviations from the random variable's expected value:

$$\sigma^2(X) = E [X - E(X)]^2.$$

The square root of variance will give us the standard deviation.

LO: Formulate an investment problem as a probability tree and explain the use of conditional expectations in investment application.

A probability tree is a means of illustrating the results of two or more independent events.

The total probability rule for expected values states unconditional expected values in terms of conditional expected values.

$$E(X|S) = P(X_1|S) X_1 + P(X_2|S) X_2 + \dots + P(X_n|S) X_n$$

LO: Calculate and interpret an updated probability in an investment setting using Bayes' formula.

Bayes' formula is a rational method for updating or adjusting the probability of an event based on new information. According to Bayes' formula, the updated probability of an event given new information is:

$$P(\text{Event} | \text{Information}) = \frac{P(\text{Information} | \text{Event})}{P(\text{Information})} \times P(\text{Event})$$