CFA Institute®
CFA Program

QUANTITATIVE METHODS

CFA® Program Curriculum
2024 • LEVEL PREREQUISITE READING • VOLUME 1

CONTENTS

O indicates an optional segment

How to Use the CFA Program Curriculum

The CFA® Program assumes basic knowledge of Economics, Quantitative Methods, and Financial Statements as presented in introductory university-level courses in Statistics, Economics, and Accounting. CFA Level I candidates who do not have a basic understanding of these concepts or would like to review these concepts can study from any of the three prerequisite reading volumes as follows:

- Prerequisite reading volume 1: Quantitative Methods
- Prerequisite reading volume 2: Economics
- Prerequisite reading volume 3: Financial Statement Analysis

ERRATA

The curriculum development process is rigorous and includes multiple rounds of reviews by content experts. Despite our efforts to produce a curriculum that is free of errors, there are instances where we must make corrections. Curriculum errata are periodically updated and posted by exam level and test date online on the Curriculum Errata webpage (www.cfainstitute.org/en/programs/submit-errata). If you believe you have found an error in the curriculum, you can submit your concerns through our curriculum errata reporting process found at the bottom of the Curriculum Errata webpage.

Quantitative Methods

LEARNING MODULE

1

Interest Rates, Present Value, and Future Value

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LEARNING OUTCOMES

Mastery	*The candidate should be able to:*
☐	interpret interest rates as required rates of return, discount rates, or opportunity costs
☐	explain an interest rate as the sum of a real risk-free rate and premiums that compensate investors for bearing distinct types of risk
☐	calculate and interpret the future value (FV) and present value (PV) of a single sum of money, an ordinary annuity, an annuity due, a perpetuity (PV only), and a series of unequal cash flows
☐	demonstrate the use of a time line in modeling and solving time value of money problems
☐	calculate the solution for time value of money problems with different frequencies of compounding
☐	calculate and interpret the effective annual rate, given the stated annual interest rate and the frequency of compounding

INTRODUCTION

1

As investment analysts, much of our work also involves evaluating transactions with present and future cash flows. When we place a value on any security, for example, we are attempting to determine the worth of a stream of future cash flows. To carry out all the above tasks accurately, we must understand the mathematics of time value of money problems. Money has time value in that individuals value a given amount of money more highly the earlier it is received. Therefore, a smaller amount of money now may be equivalent in value to a larger amount received at a future date. The

time value of money as a topic in investment mathematics deals with equivalence relationships between cash flows with different dates. Mastery of time value of money concepts and techniques is essential for investment analysts.[1]

2 INTEREST RATES

- ☐ interpret interest rates as required rates of return, discount rates, or opportunity costs
- ☐ explain an interest rate as the sum of a real risk-free rate and premiums that compensate investors for bearing distinct types of risk

In this reading, we will continually refer to interest rates. In some cases, we assume a particular value for the interest rate; in other cases, the interest rate will be the unknown quantity we seek to determine. Before turning to the mechanics of time value of money problems, we must illustrate the underlying economic concepts. In this section, we briefly explain the meaning and interpretation of interest rates.

Time value of money concerns equivalence relationships between cash flows occurring on different dates. The idea of equivalence relationships is relatively simple. Consider the following exchange: You pay $10,000 today and in return receive $9,500 today. Would you accept this arrangement? Not likely. But what if you received the $9,500 today and paid the $10,000 one year from now? Can these amounts be considered equivalent? Possibly, because a payment of $10,000 a year from now would probably be worth less to you than a payment of $10,000 today. It would be fair, therefore, to **discount** the $10,000 received in one year—that is, to cut its value based on how much time passes before the money is paid. An **interest rate**, denoted r, is a rate of return that reflects the relationship between differently dated cash flows. If $9,500 today and $10,000 in one year are equivalent in value, then $10,000 – $9,500 = $500 is the required compensation for receiving $10,000 in one year rather than now. The interest rate—the required compensation stated as a rate of return—is $500/$9,500 = 0.0526 or 5.26 percent.

Interest rates can be thought of in three ways. First, they can be considered required rates of return—that is, the minimum rate of return an investor must receive in order to accept the investment. Second, interest rates can be considered discount rates. In the example above, 5.26 percent is that rate at which we discounted the $10,000 future amount to find its value today. Thus, we use the terms "interest rate" and "discount rate" almost interchangeably. Third, interest rates can be considered opportunity costs. An **opportunity cost** is the value that investors forgo by choosing a particular course of action. In the example, if the party who supplied $9,500 had instead decided to spend it today, he would have forgone earning 5.26 percent on the money. So we can view 5.26 percent as the opportunity cost of current consumption.

Economics tells us that interest rates are set in the marketplace by the forces of supply and demand, where investors are suppliers of funds and borrowers are demanders of funds. Taking the perspective of investors in analyzing market-determined interest

1 Examples in this reading and other readings in quantitative methods were updated in 2018 by Professor Sanjiv Sabherwal of the University of Texas, Arlington.

rates, we can view an interest rate *r* as being composed of a real risk-free interest rate plus a set of four premiums that are required returns or compensation for bearing distinct types of risk:

r = Real risk-free interest rate + Inflation premium + Default risk premium + Liquidity premium + Maturity premium

- The **real risk-free interest rate** is the single-period interest rate for a completely risk-free security if no inflation were expected. In economic theory, the real risk-free rate reflects the time preferences of individuals for current versus future real consumption.
- The **inflation premium** compensates investors for expected inflation and reflects the average inflation rate expected over the maturity of the debt. Inflation reduces the purchasing power of a unit of currency—the amount of goods and services one can buy with it. The sum of the real risk-free interest rate and the inflation premium is the **nominal risk-free interest rate**.[2] Many countries have governmental short-term debt whose interest rate can be considered to represent the nominal risk-free interest rate in that country. The interest rate on a 90-day US Treasury bill (T-bill), for example, represents the nominal risk-free interest rate over that time horizon.[3] US T-bills can be bought and sold in large quantities with minimal transaction costs and are backed by the full faith and credit of the US government.
- The **default risk premium** compensates investors for the possibility that the borrower will fail to make a promised payment at the contracted time and in the contracted amount.
- The **liquidity premium** compensates investors for the risk of loss relative to an investment's fair value if the investment needs to be converted to cash quickly. US T-bills, for example, do not bear a liquidity premium because large amounts can be bought and sold without affecting their market price. Many bonds of small issuers, by contrast, trade infrequently after they are issued; the interest rate on such bonds includes a liquidity premium reflecting the relatively high costs (including the impact on price) of selling a position.
- The **maturity premium** compensates investors for the increased sensitivity of the market value of debt to a change in market interest rates as maturity is extended, in general (holding all else equal). The difference between the interest rate on longer-maturity, liquid Treasury debt and that on short-term Treasury debt reflects a positive maturity premium for the longer-term debt (and possibly different inflation premiums as well).

Using this insight into the economic meaning of interest rates, we now turn to a discussion of solving time value of money problems, starting with the future value of a single cash flow.

2 Technically, 1 plus the nominal rate equals the product of 1 plus the real rate and 1 plus the inflation rate. As a quick approximation, however, the nominal rate is equal to the real rate plus an inflation premium. In this discussion we focus on approximate additive relationships to highlight the underlying concepts.

3 Other developed countries issue securities similar to US Treasury bills. The French government issues BTFs or negotiable fixed-rate discount Treasury bills (*Bons du Trésor àtaux fixe et à intérêts précomptés*) with maturities of up to one year. The Japanese government issues a short-term Treasury bill with maturities of 6 and 12 months. The German government issues at discount both Treasury financing paper (*Finanzierungsschätze des Bundes* or, for short, *Schätze*) and Treasury discount paper (*Bubills*) with maturities up to 24 months. In the United Kingdom, the British government issues gilt-edged Treasury bills with maturities ranging from 1 to 364 days. The Canadian government bond market is closely related to the US market; Canadian Treasury bills have maturities of 3, 6, and 12 months.

3 FUTURE VALUE OF A SINGLE CASH FLOW

- ☐ calculate and interpret the future value (FV) and present value (PV) of a single sum of money, an ordinary annuity, an annuity due, a perpetuity (PV only), and a series of unequal cash flows
- ☐ demonstrate the use of a time line in modeling and solving time value of money problems

In this section, we introduce time value associated with a single cash flow or lump-sum investment. We describe the relationship between an initial investment or **present value (PV)**, which earns a rate of return (the interest rate per period) denoted as r, and its **future value (FV)**, which will be received N years or periods from today.

The following example illustrates this concept. Suppose you invest \$100 (PV = \$100) in an interest-bearing bank account paying 5 percent annually. At the end of the first year, you will have the \$100 plus the interest earned, 0.05 × \$100 = \$5, for a total of \$105. To formalize this one-period example, we define the following terms:

PV = present value of the investment

FV_N = future value of the investment N periods from today

r = rate of interest per period

For N = 1, the expression for the future value of amount PV is

$$FV_1 = PV(1 + r) \quad (1)$$

For this example, we calculate the future value one year from today as FV_1 = \$100(1.05) = \$105.

Now suppose you decide to invest the initial \$100 for two years with interest earned and credited to your account annually. At the end of the first year (the beginning of the second year), your account will have \$105, which you will leave in the bank for another year. Thus, with a beginning amount of \$105 (PV = \$105), the amount at the end of the second year will be \$105(1.05) = \$110.25. Note that the \$5.25 interest earned during the second year is 5 percent of the amount invested at the beginning of Year 2.

Another way to understand this example is to note that the amount invested at the beginning of Year 2 is composed of the original \$100 that you invested plus the \$5 interest earned during the first year. During the second year, the original principal again earns interest, as does the interest that was earned during Year 1. You can see how the original investment grows:

Original investment	\$100.00
Interest for the first year (\$100 × 0.05)	5.00
Interest for the second year based on original investment (\$100 × 0.05)	5.00
Interest for the second year based on interest earned in the first year (0.05 × \$5.00 interest on interest)	0.25
Total	\$110.25

The \$5 interest that you earned each period on the \$100 original investment is known as **simple interest** (the interest rate times the principal). **Principal** is the amount of funds originally invested. During the two-year period, you earn \$10 of simple interest. The extra \$0.25 that you have at the end of Year 2 is the interest you earned on the Year 1 interest of \$5 that you reinvested.

The interest earned on interest provides the first glimpse of the phenomenon known as **compounding**. Although the interest earned on the initial investment is important, for a given interest rate it is fixed in size from period to period. The compounded interest earned on reinvested interest is a far more powerful force because, for a given interest rate, it grows in size each period. The importance of compounding increases with the magnitude of the interest rate. For example, $100 invested today would be worth about $13,150 after 100 years if compounded annually at 5 percent, but worth more than $20 million if compounded annually over the same time period at a rate of 13 percent.

To verify the $20 million figure, we need a general formula to handle compounding for any number of periods. The following general formula relates the present value of an initial investment to its future value after *N* periods:

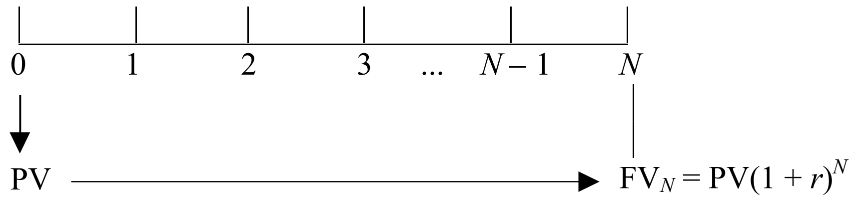
$$FV_N = PV(1 + r)^N \quad (2)$$

where *r* is the stated interest rate per period and *N* is the number of compounding periods. In the bank example, $FV_2 = \$100(1 + 0.05)^2 = \$110.25$. In the 13 percent investment example, $FV_{100} = \$100(1.13)^{100} = \$20,316,287.42$.

The most important point to remember about using the future value equation is that the stated interest rate, *r*, and the number of compounding periods, *N*, must be compatible. Both variables must be defined in the same time units. For example, if *N* is stated in months, then *r* should be the one-month interest rate, unannualized.

A time line helps us to keep track of the compatibility of time units and the interest rate per time period. In the time line, we use the time index *t* to represent a point in time a stated number of periods from today. Thus the present value is the amount available for investment today, indexed as $t = 0$. We can now refer to a time *N* periods from today as $t = N$. The time line in Exhibit 1 shows this relationship.

Exhibit 1: The Relationship between an Initial Investment, PV, and Its Future Value, FV



In Exhibit 1, we have positioned the initial investment, PV, at $t = 0$. Using Equation 2, we move the present value, PV, forward to $t = N$ by the factor $(1 + r)^N$. This factor is called a future value factor. We denote the future value on the time line as FV and position it at $t = N$. Suppose the future value is to be received exactly 10 periods from today's date ($N = 10$). The present value, PV, and the future value, FV, are separated in time through the factor $(1 + r)^{10}$.

The fact that the present value and the future value are separated in time has important consequences:

- We can add amounts of money only if they are indexed at the same point in time.
- For a given interest rate, the future value increases with the number of periods.

- For a given number of periods, the future value increases with the interest rate.

To better understand these concepts, consider three examples that illustrate how to apply the future value formula.

EXAMPLE 1

The Future Value of a Lump Sum with Interim Cash Reinvested at the Same Rate

1. You are the lucky winner of your state's lottery of $5 million after taxes. You invest your winnings in a five-year certificate of deposit (CD) at a local financial institution. The CD promises to pay 7 percent per year compounded annually. This institution also lets you reinvest the interest at that rate for the duration of the CD. How much will you have at the end of five years if your money remains invested at 7 percent for five years with no withdrawals?

Solution:

To solve this problem, compute the future value of the $5 million investment using the following values in Equation 2:

$$\text{PV} = \$5{,}000{,}000$$
$$r = 7\% = 0.07$$
$$N = 5$$
$$\text{FV}_N = \text{PV}(1 + r)^N$$
$$= \$5{,}000{,}000(1.07)^5$$
$$= \$5{,}000{,}000(1.402552)$$
$$= \$7{,}012{,}758.65$$

At the end of five years, you will have $7,012,758.65 if your money remains invested at 7 percent with no withdrawals.

In this and most examples in this reading, note that the factors are reported at six decimal places but the calculations may actually reflect greater precision. For example, the reported 1.402552 has been rounded up from 1.40255173 (the calculation is actually carried out with more than eight decimal places of precision by the calculator or spreadsheet). Our final result reflects the higher number of decimal places carried by the calculator or spreadsheet.[4]

4 We could also solve time value of money problems using tables of interest rate factors. Solutions using tabled values of interest rate factors are generally less accurate than solutions obtained using calculators or spreadsheets, so practitioners prefer calculators or spreadsheets.

EXAMPLE 2

The Future Value of a Lump Sum with No Interim Cash

1. An institution offers you the following terms for a contract: For an investment of JPY2,500,000, the institution promises to pay you a lump sum six years from now at an 8 percent annual interest rate. What future amount can you expect?

Solution:

Use the following data in Equation 2 to find the future value:

$$\begin{aligned}
\text{PV} &= ¥2,500,000\\
r &= 8\% = 0.08\\
N &= 6\\
\text{FV}_N &= \text{PV}(1+r)^N\\
&= ¥2,500,000(1.08)^6\\
&= ¥2,500,000(1.586874)\\
&= ¥3,967,186
\end{aligned}$$

You can expect to receive JPY3,967,186 six years from now.

Our third example is a more complicated future value problem that illustrates the importance of keeping track of actual calendar time.

EXAMPLE 3

The Future Value of a Lump Sum

1. A pension fund manager estimates that his corporate sponsor will make a $10 million contribution five years from now. The rate of return on plan assets has been estimated at 9 percent per year. The pension fund manager wants to calculate the future value of this contribution 15 years from now, which is the date at which the funds will be distributed to retirees. What is that future value?

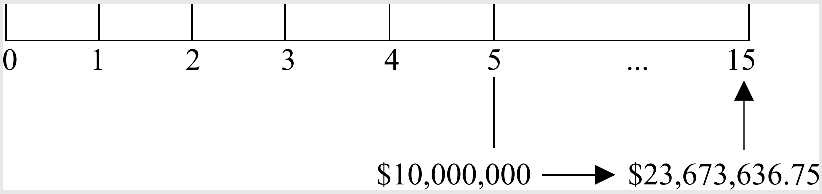
Solution:

By positioning the initial investment, PV, at t = 5, we can calculate the future value of the contribution using the following data in Equation 2:

$$\begin{aligned}
\text{PV} &= \$10 \text{ million}\\
r &= 9\% = 0.09\\
N &= 10\\
\text{FV}_N &= \text{PV}(1+r)^N\\
&= \$10,000,000(1.09)^{10}\\
&= \$10,000,000(2.367364)\\
&= \$23,673,636.75
\end{aligned}$$

This problem looks much like the previous two, but it differs in one important respect: its timing. From the standpoint of today (t = 0), the future amount of $23,673,636.75 is 15 years into the future. Although the future value is 10 years from its present value, the present value of $10 million will not be received for another five years.

Exhibit 2: The Future Value of a Lump Sum, Initial Investment Not at $t = 0$



As Exhibit 2 shows, we have followed the convention of indexing today as $t = 0$ and indexing subsequent times by adding 1 for each period. The additional contribution of $10 million is to be received in five years, so it is indexed as $t = 5$ and appears as such in the figure. The future value of the investment in 10 years is then indexed at $t = 15$—that is, 10 years following the receipt of the $10 million contribution at $t = 5$. Time lines like this one can be extremely useful when dealing with more-complicated problems, especially those involving more than one cash flow.

In a later section of this reading, we will discuss how to calculate the value today of the $10 million to be received five years from now. For the moment, we can use Equation 2. Suppose the pension fund manager in Example 3 above were to receive $6,499,313.86 today from the corporate sponsor. How much will that sum be worth at the end of five years? How much will it be worth at the end of 15 years?

$$\text{PV} = \$6{,}499{,}313.86$$
$$r = 9\% = 0.09$$
$$N = 5$$
$$\text{FV}_N = \text{PV}(1 + r)^N$$
$$= \$6{,}499{,}313.86(1.09)^5$$
$$= \$6{,}499{,}313.86(1.538624)$$
$$= \$10{,}000{,}000 \text{ at the five-year mark}$$

and

$$\text{PV} = \$6{,}499{,}313.86$$
$$r = 9\% = 0.09$$
$$N = 15$$
$$\text{FV}_N = \text{PV}(1 + r)^N$$
$$= \$6{,}499{,}313.86(1.09)^{15}$$
$$= \$6{,}499{,}313.86(3.642482)$$
$$= \$23{,}673{,}636.74 \text{ at the 15-year mark}$$

These results show that today's present value of about $6.5 million becomes $10 million after five years and $23.67 million after 15 years.

4 NON-ANNUAL COMPOUNDING (FUTURE VALUE)

☐ calculate the solution for time value of money problems with different frequencies of compounding

In this section, we examine investments paying interest more than once a year. For instance, many banks offer a monthly interest rate that compounds 12 times a year. In such an arrangement, they pay interest on interest every month. Rather than quote the periodic monthly interest rate, financial institutions often quote an annual interest rate that we refer to as the **stated annual interest rate** or **quoted interest rate**. We denote the stated annual interest rate by r_s. For instance, your bank might state that a particular CD pays 8 percent compounded monthly. The stated annual interest rate equals the monthly interest rate multiplied by 12. In this example, the monthly interest rate is 0.08/12 = 0.0067 or 0.67 percent.[5] This rate is strictly a quoting convention because $(1 + 0.0067)^{12} = 1.083$, not 1.08; the term $(1 + r_s)$ is not meant to be a future value factor when compounding is more frequent than annual.

With more than one compounding period per year, the future value formula can be expressed as

$$FV_N = PV\left(1 + \frac{r_s}{m}\right)^{mN} \tag{3}$$

where r_s is the stated annual interest rate, m is the number of compounding periods per year, and N now stands for the number of years. Note the compatibility here between the interest rate used, r_s/m, and the number of compounding periods, mN. The periodic rate, r_s/m, is the stated annual interest rate divided by the number of compounding periods per year. The number of compounding periods, mN, is the number of compounding periods in one year multiplied by the number of years. The periodic rate, r_s/m, and the number of compounding periods, mN, must be compatible.

EXAMPLE 4

The Future Value of a Lump Sum with Quarterly Compounding

1. Continuing with the CD example, suppose your bank offers you a CD with a two-year maturity, a stated annual interest rate of 8 percent compounded quarterly, and a feature allowing reinvestment of the interest at the same interest rate. You decide to invest $10,000. What will the CD be worth at maturity?

Solution:

Compute the future value with Equation 3 as follows:

$$\begin{aligned} PV &= \$10{,}000 \\ r_s &= 8\% = 0.08 \\ m &= 4 \\ r_s/m &= 0.08/4 = 0.02 \\ N &= 2 \\ mN &= 4(2) = 8 \text{ interest periods} \end{aligned}$$

$$\begin{aligned} FV_N &= PV\left(1 + \frac{r_s}{m}\right)^{mN} \\ &= \$10{,}000(1.02)^8 \\ &= \$10{,}000(1.171659) \\ &= \$11{,}716.59 \end{aligned}$$

At maturity, the CD will be worth $11,716.59.

5 To avoid rounding errors when using a financial calculator, divide 8 by 12 and then press the %*i* key, rather than simply entering 0.67 for %*i*, so we have $(1 + 0.08/12)^{12} = 1.083000$.

The future value formula in Equation 3 does not differ from the one in Equation 2. Simply keep in mind that the interest rate to use is the rate per period and the exponent is the number of interest, or compounding, periods.

EXAMPLE 5

The Future Value of a Lump Sum with Monthly Compounding

1. An Australian bank offers to pay you 6 percent compounded monthly. You decide to invest AUD 1 million for one year. What is the future value of your investment if interest payments are reinvested at 6 percent?

Solution:

Use Equation 3 to find the future value of the one-year investment as follows:

$$\text{PV} = \text{A\$}1{,}000{,}000$$
$$r_s = 6\% = 0.06$$
$$m = 12$$
$$r_s/m = 0.06/12 = 0.0050$$
$$N = 1$$
$$mN = 12(1) = 12 \text{ interest periods}$$
$$\text{FV}_N = \text{PV}\left(1 + \frac{r_s}{m}\right)^{mN}$$
$$= \text{A\$}1{,}000{,}000(1.005)^{12}$$
$$= \text{A\$}1{,}000{,}000(1.061678)$$
$$= \text{A\$}1{,}061{,}677.81$$

If you had been paid 6 percent with annual compounding, the future amount would be only AUD 1,000,000(1.06) = AUD 1,060,000 instead of AUD 1,061,677.81 with monthly compounding.

5 CONTINUOUS COMPOUNDING

- ☐ calculate and interpret the effective annual rate, given the stated annual interest rate and the frequency of compounding
- ☐ calculate the solution for time value of money problems with different frequencies of compounding

The preceding discussion on compounding periods illustrates discrete compounding, which credits interest after a discrete amount of time has elapsed. If the number of compounding periods per year becomes infinite, then interest is said to compound continuously. If we want to use the future value formula with continuous compounding, we need to find the limiting value of the future value factor for $m \to \infty$ (infinitely many compounding periods per year) in Equation 3. The expression for the future value of a sum in N years with continuous compounding is

$$\text{FV}_N = \text{PV}e^{r_s N} \tag{4}$$

The term $e^{r_s N}$ is the transcendental number $e \approx 2.7182818$ raised to the power $r_s N$. Most financial calculators have the function e^x.

EXAMPLE 6

The Future Value of a Lump Sum with Continuous Compounding

Suppose a $10,000 investment will earn 8 percent compounded continuously for two years. We can compute the future value with Equation 4 as follows:

$$\text{PV} = \$10{,}000$$
$$r_s = 8\% = 0.08$$
$$N = 2$$
$$\text{FV}_N = \text{PV}e^{r_s N}$$
$$= \$10{,}000 e^{0.08(2)}$$
$$= \$10{,}000(1.173511)$$
$$= \$11{,}735.11$$

With the same interest rate but using continuous compounding, the $10,000 investment will grow to $11,735.11 in two years, compared with $11,716.59 using quarterly compounding as shown in Example 4.

Exhibit 3 shows how a stated annual interest rate of 8 percent generates different ending dollar amounts with annual, semiannual, quarterly, monthly, daily, and continuous compounding for an initial investment of $1 (carried out to six decimal places).

As Exhibit 3 shows, all six cases have the same stated annual interest rate of 8 percent; they have different ending dollar amounts, however, because of differences in the frequency of compounding. With annual compounding, the ending amount is $1.08. More frequent compounding results in larger ending amounts. The ending dollar amount with continuous compounding is the maximum amount that can be earned with a stated annual rate of 8 percent.

Exhibit 3: The Effect of Compounding Frequency on Future Value

Frequency	r_s/m	mN	Future Value of $1		
Annual	8%/1 = 8%	1 × 1 = 1	$1.00(1.08)	=	$1.08
Semiannual	8%/2 = 4%	2 × 1 = 2	$\$1.00(1.04)^2$	=	$1.081600
Quarterly	8%/4 = 2%	4 × 1 = 4	$\$1.00(1.02)^4$	=	$1.082432
Monthly	8%/12 = 0.6667%	12 × 1 = 12	$\$1.00(1.006667)^{12}$	=	$1.083000
Daily	8%/365 = 0.0219%	365 × 1 = 365	$\$1.00(1.000219)^{365}$	=	$1.083278
Continuous			$\$1.00e^{0.08(1)}$	=	$1.083287

Exhibit 3 also shows that a \$1 investment earning 8.16 percent compounded annually grows to the same future value at the end of one year as a \$1 investment earning 8 percent compounded semiannually. This result leads us to a distinction between the stated annual interest rate and the **effective annual rate** (EAR).[6] For an 8 percent stated annual interest rate with semiannual compounding, the EAR is 8.16 percent.

Stated and Effective Rates

The stated annual interest rate does not give a future value directly, so we need a formula for the EAR. With an annual interest rate of 8 percent compounded semiannually, we receive a periodic rate of 4 percent. During the course of a year, an investment of \$1 would grow to $\$1(1.04)^2 = \$1.0816$, as illustrated in Exhibit 3. The interest earned on the \$1 investment is \$0.0816 and represents an effective annual rate of interest of 8.16 percent. The effective annual rate is calculated as follows:

$$\text{EAR} = (1 + \text{Periodic interest rate})^m - 1 \tag{5}$$

The periodic interest rate is the stated annual interest rate divided by m, where m is the number of compounding periods in one year. Using our previous example, we can solve for EAR as follows: $(1.04)^2 - 1 = 8.16$ percent.

The concept of EAR extends to continuous compounding. Suppose we have a rate of 8 percent compounded continuously. We can find the EAR in the same way as above by finding the appropriate future value factor. In this case, a \$1 investment would grow to $\$1e^{0.08(1.0)} = \$1.0833$. The interest earned for one year represents an effective annual rate of 8.33 percent and is larger than the 8.16 percent EAR with semiannual compounding because interest is compounded more frequently. With continuous compounding, we can solve for the effective annual rate as follows:

$$\text{EAR} = e^{r_s} - 1 \tag{6}$$

We can reverse the formulas for EAR with discrete and continuous compounding to find a periodic rate that corresponds to a particular effective annual rate. Suppose we want to find the appropriate periodic rate for a given effective annual rate of 8.16 percent with semiannual compounding. We can use Equation 5 to find the periodic rate:

$$\begin{aligned} 0.0816 &= (1 + \text{Periodic rate})^2 - 1 \\ 1.0816 &= (1 + \text{Periodic rate})^2 \\ (1.0816)^{1/2} - 1 &= \text{Periodic rate} \\ (1.04) - 1 &= \text{Periodic rate} \\ 4\% &= \text{Periodic rate} \end{aligned}$$

To calculate the continuously compounded rate (the stated annual interest rate with continuous compounding) corresponding to an effective annual rate of 8.33 percent, we find the interest rate that satisfies Equation 6:

$$\begin{aligned} 0.0833 &= e^{r_s} - 1 \\ 1.0833 &= e^{r_s} \end{aligned}$$

6 Among the terms used for the effective annual return on interest-bearing bank deposits are annual percentage yield (APY) in the United States and equivalent annual rate (EAR) in the United Kingdom. By contrast, the **annual percentage rate** (APR) measures the cost of borrowing expressed as a yearly rate. In the United States, the APR is calculated as a periodic rate times the number of payment periods per year and, as a result, some writers use APR as a general synonym for the stated annual interest rate. Nevertheless, APR is a term with legal connotations; its calculation follows regulatory standards that vary internationally. Therefore, "stated annual interest rate" is the preferred general term for an annual interest rate that does not account for compounding within the year.

To solve this equation, we take the natural logarithm of both sides. (Recall that the natural log of e^{r_s} is ln $e^{r_s} = r_s$.) Therefore, ln 1.0833 = r_s, resulting in r_s = 8 percent. We see that a stated annual rate of 8 percent with continuous compounding is equivalent to an EAR of 8.33 percent.

6 FUTURE VALUE OF A SERIES OF CASH FLOWS

- ☐ calculate and interpret the future value (FV) and present value (PV) of a single sum of money, an ordinary annuity, an annuity due, a perpetuity (PV only), and a series of unequal cash flows
- ☐ demonstrate the use of a time line in modeling and solving time value of money problems

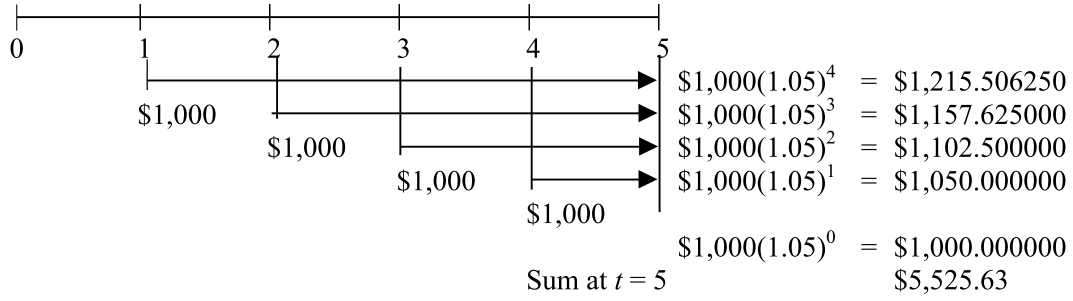
In this section, we consider series of cash flows, both even and uneven. We begin with a list of terms commonly used when valuing cash flows that are distributed over many time periods.

- An **annuity** is a finite set of level, or identical, and sequential cash flows.
- An **ordinary annuity** has a first cash flow that occurs one period from now (indexed at t = 1).
- An **annuity due** has a first cash flow that occurs immediately (indexed at t = 0).
- A **perpetuity** is a perpetual annuity, or a set of level, or identical, never-ending sequential cash flows, with the first cash flow occurring one period from now.

Equal Cash Flows—Ordinary Annuity

Consider an ordinary annuity paying 5 percent annually. Suppose we have five separate deposits of $1,000 occurring at equally spaced intervals of one year, with the first payment occurring at t = 1. Our goal is to find the future value of this ordinary annuity after the last deposit at t = 5. The increment in the time counter is one year, so the last payment occurs five years from now. As the time line in Exhibit 4 shows, we find the future value of each $1,000 deposit as of t = 5 with Equation 2, $FV_N = PV(1 + r)^N$. The arrows in Exhibit 4 extend from the payment date to t = 5. For instance, the first $1,000 deposit made at t = 1 will compound over four periods. Using Equation 2, we find that the future value of the first deposit at t = 5 is $\$1,000(1.05)^4$ = $1,215.51. We calculate the future value of all other payments in a similar fashion. (Note that we are finding the future value at t = 5, so the last payment does not earn any interest.) With all values now at t = 5, we can add the future values to arrive at the future value of the annuity. This amount is $5,525.63.

Exhibit 4: The Future Value of a Five-Year Ordinary Annuity



We can arrive at a general annuity formula if we define the annuity amount as A, the number of time periods as N, and the interest rate per period as r. We can then define the future value as

$$FV_N = A\left[(1+r)^{N-1} + (1+r)^{N-2} + (1+r)^{N-3} + \ldots + (1+r)^1 + (1+r)^0\right]$$

which simplifies to

$$FV_N = A\left[\frac{(1+r)^N - 1}{r}\right] \tag{7}$$

The term in brackets is the future value annuity factor. This factor gives the future value of an ordinary annuity of $1 per period. Multiplying the future value annuity factor by the annuity amount gives the future value of an ordinary annuity. For the ordinary annuity in Exhibit 4, we find the future value annuity factor from Equation 7 as

$$\left[\frac{(1.05)^5 - 1}{0.05}\right] = 5.525631$$

With an annuity amount A = $1,000, the future value of the annuity is $1,000(5.525631) = $5,525.63, an amount that agrees with our earlier work.

The next example illustrates how to find the future value of an ordinary annuity using the formula in Equation 7.

EXAMPLE 7

The Future Value of an Annuity

1. Suppose your company's defined contribution retirement plan allows you to invest up to EUR20,000 per year. You plan to invest €20,000 per year in a stock index fund for the next 30 years. Historically, this fund has earned 9 percent per year on average. Assuming that you actually earn 9 percent a

year, how much money will you have available for retirement after making the last payment?

Solution:

Use Equation 7 to find the future amount:

A = €20,000

$r = 9\% = 0.09$

$N = 30$

$$\text{FV annuity factor} = \frac{(1+r)^N - 1}{r} = \frac{(1.09)^{30} - 1}{0.09} = 136.307539$$

FV_N = €20,000(136.307539)

= €2,726,150.77

Assuming the fund continues to earn an average of 9 percent per year, you will have €2,726,150.77 available at retirement.

Unequal Cash Flows

In many cases, cash flow streams are unequal, precluding the simple use of the future value annuity factor. For instance, an individual investor might have a savings plan that involves unequal cash payments depending on the month of the year or lower savings during a planned vacation. One can always find the future value of a series of unequal cash flows by compounding the cash flows one at a time. Suppose you have the five cash flows described in Exhibit 5, indexed relative to the present (t = 0).

Exhibit 5: A Series of Unequal Cash Flows and Their Future Values at 5 Percent

Time	Cash Flow ($)	Future Value at Year 5		
t = 1	1,000	$1,000(1.05)^4	=	$1,215.51
t = 2	2,000	$2,000(1.05)^3	=	$2,315.25
t = 3	4,000	$4,000(1.05)^2	=	$4,410.00
t = 4	5,000	$5,000(1.05)^1	=	$5,250.00
t = 5	6,000	$6,000(1.05)^0	=	$6,000.00
		Sum	=	$19,190.76

All of the payments shown in Exhibit 5 are different. Therefore, the most direct approach to finding the future value at t = 5 is to compute the future value of each payment as of t = 5 and then sum the individual future values. The total future value at Year 5 equals $19,190.76, as shown in the third column. Later in this reading, you will learn shortcuts to take when the cash flows are close to even; these shortcuts will allow you to combine annuity and single-period calculations.